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On the number of edges of cyclic subgroup graphs of finite groups

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Abstract. In this note, we show that among finite nilpotent groups of a given order or finite groups of a given odd order, the cyclic group of that order has the minimum number of edges in its cyclic subgroup graph. We also conjecture that this holds for arbitrary finite groups.

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1. Introduction. Recently, mathematicians constructed many graphs which are assigned to groups by different methods (see, e.g., [4]). One of them is the *subgroup graph* $L(G)^*$ of a group G, that is, the graph whose vertices are the subgroups of G and two vertices H_1 and H_2 are connected by an edge if and only if $H_1 \leq H_2$ and there is no subgroup $K \leq G$ such that $H_1 < K < H_2$. Finite groups with planar subgroup graph have been classified by Starr and Turner [14], Bohanon and Reid [3], and Schmidt [13]. Also, the genus of $L(G)^*$ has been studied by Lucchini [10].

In the current note, we will focus on a remarkable subgraph of $L(G)^*$, namely the *cyclic subgroup graph* $C(G)^*$ of G. Its vertex set is the poset $C(G)$ of cyclic subgroups of G and, similarly, two vertices H_1 and H_2 are connected by an edge if and only if $H_1 \leq H_2$ and $H_1 < K < H_2$ for no cyclic subgroup K of G. Note that the (cyclic) subgroup graph of a group G is the Hasse diagram of the poset of (cyclic) subgroups of G, viewed as a simple, undirected graph.

We will prove that if G is a nilpotent group of order n or a group of odd order n, then the minimum number of edges of $C(G)^*$ is attained for the cyclic group $\mathbb{Z}_n$. This fact was somewhat expected since finite group theory abounds with functions attaining their minimum/maximum at cyclic groups, such as the function sum of element orders [1], the function number of (cyclic) subgroups [6,7,12], or the function number of edges of (directed) power graphs [5].

Our main result is as follows.

Theorem 1.1. *Let G be a finite group of order n. If G is nilpotent or n is odd, then*

$$|E(C(G)^*)| \geq |E(C(\mathbb{Z}_n)^*)|.$$

Moreover, we have equality if and only if $G \cong \mathbb{Z}_n$.

For the proof of Theorem 1.1, we need the following theorem.

Theorem. *Let G be a finite group of order n. Then there is a bijection $f : G \longrightarrow \mathbb{Z}_n$ such that $o(a)$ divides $o(f(a))$ for all $a \in G$.*

This has been formulated as a question by Isaacs (see [11, Problem 18.1]) and proved for solvable groups by Ladisch [9]. A proof for arbitrary groups has been recently proposed by Amiri [2]. We point out that we need this bijection only for groups of odd order, which are solvable.

We conjecture that the inequality $|E(C(G)^*)| \geq |E(C(\mathbb{Z}_n)^*)|$ also holds for any group G of even order n. Note that in this case there are examples of non-cyclic groups G of order n with $|E(C(G)^*)| = |E(C(\mathbb{Z}_n)^*)|$, such as

$$|E(C(A_4)^*)| = |E(C(\text{Dic}_3)^*)| = |E(C(\mathbb{Z}_{12})^*| = 7.$$

Most of our notation is standard and will usually not be repeated here. For basic notions and results on groups, we refer the reader to [8].

2. Proof of the main result. We start by proving some auxiliary results.

Lemma 2.1. *Let G_i, $i = 1, ..., k$, be finite groups of coprime orders and $G = G_1 \times \cdots \times G_k$. Then*

$$|E(C(G)^*)| = \left(\sum_{i=1}^{k} \frac{|E(C(G_i)^*)|}{|C(G_i)|}\right) \prod_{i=1}^{k} |C(G_i)|. \tag{1}$$

In particular, if $n = p_1^{n_1} \cdots p_k^{n_k}$ is the decomposition of $n \in \mathbb{N}^*$ as a product of prime factors, then

$$|E(C(\mathbb{Z}_n)^*)| = \left(\sum_{i=1}^{k} \frac{n_i}{n_i + 1}\right) \prod_{i=1}^{k} (n_i + 1). \tag{2}$$

Proof. The equality (1) follows immediately by induction on k. For the equality (2), it suffices to observe that $|C(\mathbb{Z}_{p_i^{n_i}})| = n_i + 1$ and $|E(C(\mathbb{Z}_{p_i^{n_i}})^*)| = n_i$ for all $i = 1, ..., k$. □

A simple way to count the number of edges in the cyclic subgroup graph of a finite group is given by the following lemma.

Lemma 2.2. *Let G be a finite group. Then*

$$|E(C(G)^*)| = \sum_{a \in G} \frac{\omega(o(a))}{\varphi(o(a))}, \tag{3}$$

where $\omega(d)$ is the number of distinct primes dividing $d \in \mathbb{N}^*$ and φ is the Euler totient function.

Proof. We have
$$|E(C(G)^*)| = \sum_{H \in C(G)} |\mathrm{Max}(H)|,$$
where $\mathrm{Max}(H)$ denotes the set of maximal subgroups of $H \in C(G)$. Clearly, this leads to (3) because a cyclic subgroup $H = \langle a \rangle$ of G has $\omega(o(a))$ maximal subgroups and $\varphi(o(a))$ generators. $\square$

The next lemma shows that the function in the right side of (3) is decreasing with respect to divisibility on the set of odd positive integers.

Lemma 2.3. *Let d and d' be two odd positive integers such that $d \mid d'$ and $d \geq 3$. Then*
$$\frac{\omega(d)}{\varphi(d)} \geq \frac{\omega(d')}{\varphi(d')}. \tag{4}$$
Moreover, we have equality if and only if either $d = d'$ or d is a prime power p^α with $p \geq 5$ and $d' = 3d$.

Proof. Let $d = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $d' = p_1^{\beta_1} \cdots p_r^{\beta_r} p_{r+1}^{\beta_{r+1}} \cdots p_s^{\beta_s}$, where $1 \leq \alpha_i \leq \beta_i$ for all $i = 1, ..., r$ and $r \leq s$. Then (4) becomes
$$p_1^{\beta_1-\alpha_1} \cdots p_r^{\beta_r-\alpha_r} p_{r+1}^{\beta_{r+1}-1} \cdots p_s^{\beta_s-1} (p_{r+1}-1) \cdots (p_s-1) \geq \frac{s}{r},$$
which is true for $s = r$. So, we can assume that $s \geq r+1$. We remark that it suffices to show
$$(p_{r+1}-1) \cdots (p_s-1) \geq \frac{s}{r}.$$
Since each p_i is odd, we get
$$(p_{r+1}-1) \cdots (p_s-1) \geq 2^{s-r}.$$
On the other hand, we can easily see that the function $f(x) = 2^{x-r} - \frac{x}{r}$ is increasing for $x \geq r+1$. Thus
$$2^{s-r} - \frac{s}{r} \geq 2 - \frac{r+1}{r} \geq 0,$$
as desired.

Note that the equality occurs in (4) if and only if either
$$r = s \text{ and } \alpha_i = \beta_i, \forall\, i = 1, ..., r,$$
or
$$r = 1, s = 2, \alpha_1 = \beta_1, \beta_2 = 1, \text{ and } p_2 = 3,$$
that is, if and only if either $d = d'$ or $d = p_1^{\alpha_1}$ with $p_1 \geq 5$ and $d' = 3d$. This completes the proof. $\square$

We mention that the inequality (4) is not true when d' is even. For example, for d odd and $d' = 2d$, we have
$$\frac{\omega(d)}{\varphi(d)} < \frac{\omega(d)+1}{\varphi(d)} = \frac{\omega(d')}{\varphi(d')}.$$

We are now able to prove our main result.

Proof of Theorem 1.1. Assume first that G is nilpotent and let $G = G_1 \times \cdots \times G_k$ be its decomposition as a direct product of Sylow subgroups. Denote $|G_i| = p_i^{n_i}$, $i = 1, ..., k$. By [7, Theorem 2.5], we have

$$|C(G_i)| \geq |C(\mathbb{Z}_{p_i^{n_i}})| = n_i + 1,$$

with equality if and only if $G_i \cong \mathbb{Z}_{p_i^{n_i}}$. Using Lemma 2.2, this leads to

$$\begin{aligned}|E(C(G_i)^*)| &= \sum_{H \in C(G_i)} |\mathrm{Max}(H)| = \sum_{H \in C(G_i) \setminus 1} 1 = |C(G_i)| - 1 \\ &\geq |C(\mathbb{Z}_{p_i^{n_i}})| - 1 = |E(C(\mathbb{Z}_{p_i^{n_i}})^*)| = n_i.\end{aligned}$$

Then equalities (1) and (2) in Lemma 2.1 imply that

$$|E(C(G)^*)| \geq |E(C(\mathbb{Z}_n)^*)|,$$

with equality if and only if $G \cong \mathbb{Z}_n$.

Assume next that n is odd and let $f : G \longrightarrow \mathbb{Z}_n$ be a bijection such that $o(a)$ divides $o(f(a))$ for all $a \in G$. Then

$$\frac{\omega(o(a))}{\varphi(o(a))} \geq \frac{\omega(o(f(a)))}{\varphi(o(f(a)))}, \forall\, a \in G,$$

by Lemma 2.3 and therefore we have

$$|E(C(G)^*)| = \sum_{a \in G} \frac{\omega(o(a))}{\varphi(o(a))} \geq \sum_{a \in G} \frac{\omega(o(f(a)))}{\varphi(o(f(a)))} = |E(C(\mathbb{Z}_n)^*)|.$$

Finally, assume that $|E(C(G)^*)| = |E(C(\mathbb{Z}_n)^*)|$, but G is not cyclic. Then

$$\frac{\omega(o(a))}{\varphi(o(a))} = \frac{\omega(o(f(a)))}{\varphi(o(f(a)))}, \forall\, a \in G.$$

Let $a_1, ..., a_{\varphi(n)} \in G$ with $o(f(a_i)) = n$ for all i. By Lemma 2.3, it follows that there is a prime p and a positive integer α such that

$$n = 3p^\alpha \text{ and } o(a_i) = p^\alpha, \forall\, i = 1, ..., \varphi(n).$$

Then, from Sylow's theorems, we infer that G has a unique (normal) Sylow p-subgroup. On the other hand, since G contains at least $\varphi(n) = 2\varphi(p^\alpha)$ elements of order p^α, it will contain at least two cyclic subgroups of order p^α, a contradiction.

The proof of Theorem 1.1 is now complete. □

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